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explanation given to them and the method of finding and proving them. Nor will this method, so far as I can see, while making them intelligible and bringing them within the easy comprehension of all persons, put the learner in any position of disadvantage in reference to the higher and more difficult questions that he must encounter in his further pursuit of knowledge or in the practical applications of it to the various purposes of life.

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ON THE DIFFERENTIAL EQUATIONS OF DYNAMICS.

BY G. W. HILL.

The general formula of dynamics is

$$\Sigma\left[\left(m\frac{d^2x}{dt^2}-\mathrm{X}\right)\delta x+\left(m\frac{d^2y}{dt^2}-\mathrm{Y}\right)\delta y+\left(m\frac{d^2z}{dt^2}-\mathrm{Z}\right)\delta z\right]=0.$$

In the usual treatment of this equation, we have been asked to attribute to the symbols δx, δy, δz, &c., the signification they have in the calculus of variations. This however is unnecessary, except when we wish to deduce from it the principle of least action; and the student, unacquainted with this calculus, may regard these symbols as multipliers, which, when all the points of the system are free, have any finite values we please, but when the coordinates are restricted to satisfy an equation $\mathrm{U} = 0$, are subject to the condition

$$\frac{d\mathrm{U}}{dx}\,\delta x+\frac{d\mathrm{U}}{dy}\,\delta y+\frac{d\mathrm{U}}{dz}\,\delta z+\text{\&c.}=0,$$

an equation which, for brevity, we shall write $\delta\mathrm{U} = 0$.

We shall confine our attention to those cases in which the equations of condition and the accelerating forces are functions of the coordinates and the time only, and in which the latter are equivalent to the partial differential coefficients of a single function Ω taken with respect to the coordinates whose acceleration they express.

Whenever a function as U involves, in addition to x, y, z, &c., their first differential coefficients with respect to the time, quantities which we shall denote by x', y', z', &c., we shall suppose that $\delta\mathrm{U}$ involves, besides the terms written above, the following

$$\frac{dU}{dx'}\delta x' + \frac{dU}{dy'}\delta y' + \frac{dU}{dz'}\delta z' + \&c.$$

Moreover as we shall have to differentiate such functions as δU with respect to t, we shall meet with such quantities as $\frac{d\delta x}{dt}$, and shall suppose that the order of the symbols d and δ may be inverted, that is, we shall have equations such as

$$\frac{d\delta x}{dt} = \delta\frac{dx}{dt} = \delta x'.$$

The reader will see in this only a notational assumption, without quantitative significance, serving merely as machinery of demonstration. It will be noted that t is a variable not subject to the operation δ.

We have

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = \delta\Omega,$$

and for convenience may put

$$\tfrac{1}{2}\Sigma m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right) = T.$$

Then it will readily be perceived that the general formula can bc written thus

$$\frac{d}{dt}.\Sigma m\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right) - \delta(T + \Omega) = 0.$$

The coordinates x, y, z, &c., can be expressed as functions of the time and certain variables q_i, independent of each other and whose number is equal to that of the variables x, y, z, &c. diminished by the number of equations of condition. Substituting for x, y, z, &c. their values in terms of the new variables q_i, it is plain that the last equation will take the following form,

$$\frac{d}{dt}.\Sigma_i p_i\delta q_i - \delta(T + \Omega) = 0.$$

We can find the value of p_i, without actually making the substitution, from this consideration; since the original equation contains only the variations δx, δy, δz, &c. without the variations $\delta\frac{dx}{dt}$, $\delta\frac{dy}{dt}$ $\delta\frac{dz}{dt}$ &c., it follows that, in its transformed state, it should contain only the variations δq_i without the variations $\delta\frac{dq_i}{dt}$.

Then writing q_i' for $\frac{dq_i}{dt}$, the coefficient of $\delta q_i'$ should vanish in the equation

$$\Sigma_i\left(\frac{dp_i}{dt}\delta q_i + p_i\delta q_i'\right) - \delta(\mathrm{T} + \Omega) = 0.$$

That is, since Ω does not contain q_i',

$$p_i = \frac{d\mathrm{T}}{dq_i'}.$$

Thus the general formula becomes

$$\frac{d}{dt}.\,\Sigma_i\left(\frac{d\mathrm{T}}{dq_i'}\,\delta q_i\right) - \delta(\mathrm{T} + \Omega) = 0.$$

Because in this equation the variables q_i are independent, we may equate the coefficient of each δq_i to zero. Thus

$$\frac{d}{dt}.\frac{d\mathrm{T}}{dq_i'} - \frac{d(\mathrm{T} + \Omega)}{dq_i} = 0.$$

This is Lagrange's canonical form of the differential equations of motion.

A simpler form may be obtained by substituting the variables p_i for q_i'. By adding to and subtracting from the general formula, the term $\delta.\Sigma_i(p_i q_i')$, and writing

$$\mathrm{H} = \Sigma_i(p_i q_i') - \mathrm{T} - \Omega,$$

it becomes

$$\Sigma_i\left(\frac{dp_i}{dt}\,\delta q_i - \frac{dq_i}{dt}\,\delta p_i\right) + \delta\mathrm{H} = 0.$$

Equating the coefficients of each variation δq_i and δp_i to zero gives the equations

$$\frac{dp_i}{dt} = -\frac{d\mathrm{H}}{dq_i}, \qquad \frac{dq_i}{dt} = \frac{d\mathrm{H}}{dp_i},$$

which are known as Hamilton's canonical form.

The expression for H can take a simpler shape. From the value of T, it is evident that a certain part of it is independent of the variables q_i, which may be denoted by T_0, another part T_1, involves the first powers, and a third T_2 involves the squares and products of the same; then $T = T_0 + T_1 + T_2$. By the theory of homogeneous functions

$$\Sigma_i(p_i q_i') = \Sigma_i\left(\frac{dT}{dq_i'} q_i'\right) = T_1 + 2T_2.$$

Hence, writing

$$\Omega' = \Omega + T_0,$$
$$H = T_2 - \Omega'.$$

GEOMETRICAL PROBLEM.

BY T. P STOWELL, ROCHESTER, N, Y.

In any triangle whose angles are all known, as also the distances from the three angles to any point either within or without the triangle, the sides may be found as follows:

Let ABC be the triangle, and P a point whose distances PA, PB and PC are given; revolve the triangle APC over the side AC, P will be at F; and APB revolved over AB, P will be at D; also BPC revolved over BC, P will be at E; then join DE, EF and FD.

The △AFD is known, having two equal sides, AF and AD, and the included angle DAF double the angle BAC. Hence DF is known. Similarly FE and DE are known, and therefore all the angles of the △ DEF are known. Consequently ∠ APC = AFD + DFE + CFE, and therefore the sides AC, AB and BC are easily found. This question admits of an easy geometrical construction.